

# Casimir Stress by Euler-Maclaurin Summation

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(August 3, 2021)

We calculate the Casimir energy and pressure due to quantum fluctuations of the electromagnetic field in vacuo. We accomplish renormalisation by utilising the Euler-Maclaurin summation formula.

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## 1. The Euler-Maclaurin Summation Formula

**Definition.** Given a function  $f$  of (at least) class  $C^m$  ( $m \in \mathbb{Z}^+$ ), the Euler-Maclaurin summation formula is a mathematical identity relating the sum of  $f$  to its integral and is given by:

$$\sum_{k=n}^N f(k) - \int_n^N f(x) dx = \frac{1}{2} \left( f(n) + f(N) \right) + \sum_{r=1}^m \frac{B_{2r}}{(2r)!} \left( f^{(2r-1)}(N) - f^{(2r-1)}(n) \right) + \frac{1}{(2m+1)!} \int_n^N P_{2m+1}(x) f^{(2m+1)}(x) dx \quad (1)$$

where  $\{B_k\}$  denote the set of Bernoulli numbers,  $\{P_k(x)\} = \{B_k(x - [x])\}$  denote the set of associated periodic Bernoulli polynomials and  $n, N \in \mathbb{N}$ .

## 2. The Casimir Energy

The spectrum of modes within a perfectly conducting box of dimensions  $L_x \times L_y \times L_z$  with vacuum interior is given

$$\frac{\omega_N}{c_0} = \sqrt{\frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2}},$$

where  $n_x, n_y, n_z \in \mathbb{Z}^+$  and  $N \equiv n_x, n_y, n_z$ . We introduce the non-dimensional  $\Omega_N$  by

$$\Omega_N \equiv \frac{\omega_N L_z}{c_0} = \pi \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}, \quad (2)$$

The vacuum energy of the system is then a sum over all possible modes:

$$\mathcal{E} = \frac{\hbar}{2} \sum_N \omega_N = \frac{\hbar c_0}{2L_z} \sum_N \Omega_N.$$

This expression is infinite, so must be regularised:

$$\mathcal{E}(s) = \frac{\hbar c_0}{2L_z} \sum_N \Omega_N W_s(\Omega_N) = \frac{\hbar \pi c_0}{2L_z} \sum_N \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2} W_s \left( \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2} \right).$$

in terms of the regulator  $W_s(\Omega_N)$  which is a smooth function of  $s \geq 0$  and  $\Omega_N$  with  $W_0(\Omega_N) = 1$  for all  $\Omega_N$ . To analyse the geometry of the Casimir effect, we consider the limit  $L_x, L_y \rightarrow \infty$ , so that  $n_x, n_y$  become continuous and we may write

$$\mathcal{E}(s) = \frac{\hbar \pi c_0}{2L_z} \sum_{n_z=0}^{\infty} \int_{n_x=0}^{\infty} dn_x \int_{n_y=0}^{\infty} dn_y \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2} W_s \left( \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2} \right).$$

Introducing the notation

$$r_x = \frac{n_x L_z}{L_x} \quad \text{and} \quad r_y = \frac{n_y L_z}{L_y}, \quad (3)$$

we may write this as

$$\mathcal{E}(s) = L_x L_y \cdot \frac{\hbar \pi c_0}{2L_z^3} \sum_{n_z=0}^{\infty} \int_{n_x=0}^{\infty} dr_x \int_{n_y=0}^{\infty} dr_y \sqrt{r_x^2 + r_y^2 + n_z^2} W_s \left( \sqrt{r_x^2 + r_y^2 + n_z^2} \right).$$

We may change variables using a polar coordinate representation ( $r_x = R \sin \theta, r_y = R \cos \theta$ ) with  $R^2 = r_x^2 + r_y^2$  and  $dr_x dr_y = R dR d\theta$  so that

$$\begin{aligned} \frac{\mathcal{E}}{L_x L_y} &= \frac{\hbar \pi c_0}{2L_z^3} \sum_{n_z=0}^{\infty} \int_{R=0}^{\infty} \int_{\theta=0}^{\pi/2} R dR d\theta \sqrt{R^2 + n_z^2} W_s \left( \sqrt{R^2 + n_z^2} \right) \\ &= \frac{\hbar \pi^2 c_0}{4L_z^3} \sum_{n_z=0}^{\infty} \int_{R=0}^{\infty} R dR \sqrt{R^2 + n_z^2} W_s \left( \sqrt{R^2 + n_z^2} \right) \end{aligned}$$

Introducing the change of variable  $U^2 = R^2 + n_z^2$  with  $U dU = R dR$ , we have

$$\frac{\mathcal{E}}{L_x L_y} = \frac{\hbar \pi^2 c_0}{4L_z^3} \sum_{n_z=0}^{\infty} \int_{U=n_z}^{\infty} U^2 W_s(U) dU = \frac{\hbar \pi^2 c_0}{4L_z^3} \sum_{n_z=0}^{\infty} f_s(n_z),$$

where

$$f_s(n_z) \equiv \int_{U=n_z}^{\infty} U^2 W_s(U) dU.$$

If we choose the regulator  $W_s(U) = e^{-sU}$  then

$$f_s(n_z) = \int_{U=n_z}^{\infty} U^2 e^{-sU} dU$$

We may then use the Euler-Maclaurin summation formula (1) in order to evaluate  $\mathcal{E}(s)$ . With  $m = 2$ , we have

$$\begin{aligned} \sum_{n_z=0}^{\infty} f_s(n_z) - \int_0^{\infty} f_s(x) dx &= \frac{1}{2} \left( f_s(0) + f_s(\infty) \right) + \frac{B_2}{2} (f'_s(\infty) - f'_s(0)) + \frac{B_4}{24} (f'''_s(\infty) - f'''_s(0)) \\ &\quad + \frac{1}{120} \int_0^{\infty} P_5(x) f_s^{(5)}(x) dx \\ &= \frac{1}{2} \int_0^{\infty} U^2 e^{-sU} dU - \frac{1}{360} + O(s^2) \\ &= \frac{1}{s^3} - \frac{1}{360} + O(s^2) \end{aligned}$$

using

$$\frac{d}{da} f_s(a) = \frac{d}{da} \int_a^{\infty} U^2 e^{-sU} dU = -a^2 e^{-sa}$$

and

$$\begin{aligned} \int_0^{\infty} U^2 e^{-sU} dU &= -\frac{U^2}{s} e^{-sU} \Big|_{U=0}^{\infty} + \frac{2}{s} \int_{U=n_z}^{\infty} U e^{-sU} dU \\ &= \frac{2}{s} \left( -\frac{U}{s} e^{-sU} \Big|_{U=0}^{\infty} + \frac{1}{s} \int_{U=n_z}^{\infty} e^{-sU} dU \right) \\ &= \frac{2}{s^2} \left( \frac{e^{-sU}}{s} \Big|_{U=n_z}^{\infty} \right) = \frac{2}{s^2} \cdot \frac{1}{s} = \frac{2}{s^3}. \end{aligned}$$

The terms of order  $s^2$  occur from the integral involving the periodic Bernoulli functions and so will vanish upon renormalization. Thus, we define the regularized energy density

$$\frac{\mathcal{E}_{\text{reg}}(s)}{L_x L_y} = \frac{\hbar \pi^2 c_0}{4L_z^3} \left( \sum_{n_z=0}^{\infty} f_s(n_z) - \int_0^{\infty} f_s(x) dx - \frac{1}{s^3} \right)$$

and the renormalized energy density is

$$\mathcal{U}_{\text{ren}} \equiv \frac{\mathcal{E}_{\text{reg}}(0)}{L_x L_y} = -\frac{\hbar \pi^2 c_0}{1440 L_z^3}.$$

This is the renormalized energy density for one of the modes. There is an equal energy density for the other mode and so the total energy density is simply

$$\mathcal{U}_{\text{TOT}} = 2\mathcal{U}_{\text{ren}} = -\frac{\hbar \pi^2 c_0}{720 L_z^3}.$$

### 3. The Casimir Pressure

In this section, we shall consider only the TE modes (an analogous approach works for the TM modes also). The pre-potential for the TE modes in our perfectly conducting box are given by

$$\phi_N^{\text{TE}} = \mathcal{N}_N^{\text{TE}} \cos(k_x x) \cos(k_y y) \sin(k_z z) dz \quad (4)$$

in terms of the normalization constant  $\mathcal{N}_N^{\text{TE}}$  and where

$$k_x = \frac{\pi n_x}{L_x}, \quad k_y = \frac{\pi n_y}{L_y} \quad \text{and} \quad k_z = \frac{\pi n_z}{L_z}.$$

Given the pre-potential, a mode of the TE component of the (spatial) potential 1-form is given by

$$A_N^{\text{TE}} = \#d\phi_N^{\text{TE}}$$

yielding a mode of the operator-valued potential 1-form

$$\hat{A}_N^{\text{TE}} = \hat{a}_N A_N^{\text{TE}} e^{i\omega_N t} + \hat{a}_N^\dagger \overline{A_N^{\text{TE}}} e^{-i\omega_N t} = \left( \hat{a}_N e^{i\omega_N t} + \hat{a}_N^\dagger e^{-i\omega_N t} \right) \#d\phi_N^{\text{TE}}$$

since  $A_N^{\text{TE}}$  is real and we have introduced the creation and annihilation operators  $\hat{a}_N^\dagger, \hat{a}_N$  respectively satisfying

$$\left[ \hat{a}_N, \hat{a}_M^\dagger \right] = \delta_{NM} \quad \text{and} \quad \hat{a}_N |0\rangle = 0 \quad (5)$$

where  $|0\rangle$  denotes the vacuum state with  $\langle 0|0\rangle = 1$ . The fields are obtained from  $\hat{A}_N^{\text{TE}}$ :

$$\hat{e}_N^{\text{TE}} = i\omega_N \hat{A}_N^{\text{TE}} \quad \text{and} \quad \hat{b}_N^{\text{TE}} = \#d\hat{A}_N^{\text{TE}}.$$

In order to normalize the modes, we equate the vacuum expectation value of the energy to  $1/2\hbar\omega_N$  on a mode-by-mode basis. The vacuum energy is given by

$$E_N \equiv \int_{\mathcal{V}} \langle 0 | \hat{E}_{NM} | 0 \rangle$$

where  $\mathcal{V}$  denotes the volume of the perfectly conducting box and in terms of the operator-valued energy density 3-form

$$\hat{E}_{NM} \equiv \frac{\epsilon_0}{2} \left( \hat{e}_N^{\text{TE}} \wedge \# \overline{\hat{e}_M^{\text{TE}}} + c_0^2 \hat{b}_N^{\text{TE}} \wedge \# \overline{\hat{b}_M^{\text{TE}}} \right).$$

The fields in the perfectly conducting box for the TE modes are given using the pre-potential  $\phi_N^{\text{TE}}$  (4) and yield the vacuum expectation of the operator-valued energy density 3-form<sup>1</sup>

$$\langle 0 | \hat{E}_{NM} | 0 \rangle = \frac{\epsilon_0}{2} \left( \omega_N^2 A_N^{\text{TE}} \wedge \# A_N^{\text{TE}} + c_0^2 \#dA_N^{\text{TE}} \wedge dA_N^{\text{TE}} \right).$$

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<sup>1</sup>See (A) for further details of vacuum expectation values

Integrating this over the whole box  $\mathcal{V}$  we obtain

$$\begin{aligned} E_N &= \int_{\mathcal{V}} \langle 0 | \hat{E}_{NM} | 0 \rangle = \frac{1}{8} \epsilon_0 (\mathcal{N}_N^{\text{TE}})^2 c_0^2 L_x L_y L_z (k_x^2 + k_y^2) (k_x^2 + k_y^2 + k_z^2) \\ &= \frac{1}{8} \epsilon_0 (\mathcal{N}_N^{\text{TE}})^2 L_x L_y L_z (k_x^2 + k_y^2) \omega_N^2. \end{aligned}$$

Thus, by equating this to  $1/2 \hbar \omega_N$

$$E_N = \int_{\mathcal{V}} \langle 0 | \hat{E}_{NM} | 0 \rangle = \frac{1}{2} \hbar \omega_N$$

we are able to normalize the modes. We have

$$(\mathcal{N}_N^{\text{TE}})^2 = \frac{4\hbar}{\epsilon_0 L_x L_y L_z (k_x^2 + k_y^2) \omega_N}. \quad (6)$$

Having normalized the modes, we may now calculate the force at a face of the box by using the operator-valued stress 2-forms

$$\sigma_{NM}^K = -\frac{\epsilon_0}{2} \left( \hat{e}_N^{\text{TE}}(K) \wedge \# \overline{\hat{e}_M^{\text{TE}}} + c^2 \# \hat{b}_N^{\text{TE}} \wedge \overline{\hat{b}_M^{\text{TE}}}(K) + \hat{e}_N^{\text{TE}} \wedge i_K \# \overline{\hat{e}_M^{\text{TE}}} - c^2 i_K \# \hat{b}_N^{\text{TE}} \wedge \overline{\hat{b}_M^{\text{TE}}} \right)$$

in terms of a Killing vector  $K$ . The vacuum expectation value of the force component in the  $K$ -direction due to one mode of the quantum electromagnetic fluctuations in the vacuum is given by

$$\mathcal{F}_N^K = \int_{\Sigma} \langle 0 | \sigma_{NM}^K | 0 \rangle.$$

The vacuum expectation value of the total force is then given by summing all these modes:

$$\mathcal{F}^K = \sum_N \mathcal{F}_N^K.$$

To compute the vacuum expectation value of the force on the  $xy$ -face at  $z = L_z$  we use the Killing vector  $K = \partial_z$ . Using the fields as defined above, we obtain

$$\mathcal{F}_N^K = \frac{\epsilon_0 (\mathcal{N}_N^{\text{TE}})^2 c^2 \pi^2 n_z^2 (k_x^2 + k_y^2) L_x L_y}{8L_z^2}.$$

Substituting in for the normalization coefficient  $\mathcal{N}_N^{\text{TE}}$  from (6) yields

$$\mathcal{F}_N^K = \frac{\hbar \pi^2 c_0^2 n_z^2}{2L_z^3 \omega_N} = \frac{\hbar \pi^2 c_0 n_z^2}{2L_z^2 \Omega_N}.$$

using (2). Thus, the vacuum expectation value of the total force is given by

$$\mathcal{F}^K = \frac{\hbar \pi^2 c_0}{2L_z^2} \sum_N \frac{n_z^2}{\Omega_N}.$$

Once again this expression is infinite so needs to be regularized:

$$\mathcal{F}^K(s) = \frac{\hbar\pi^2 c_0}{2L_z^2} \sum_N \frac{n_z^2}{\Omega_N} W_s(\Omega_N)$$

in terms of some regulating function  $W_s(\Omega_N)$ . Expanding this out we have

$$\mathcal{F}^K(s) = \frac{\hbar\pi c_0}{2L_z^2} \sum_N \frac{n_z^2}{\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}} W_s \left( \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2} \right)$$

using (2). As before, we consider the limit  $L_x, L_y \rightarrow \infty$  so that  $n_x, n_y$  become continuous and we may write

$$\mathcal{F}^K(s) = \frac{\hbar\pi c_0}{2L_z^2} \sum_{n_z=0}^{\infty} \int_{n_x=0}^{\infty} dn_x \int_{n_y=0}^{\infty} dn_y \frac{n_z^2}{\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}} W_s \left( \sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2} \right)$$

Using the  $r_x, r_y$  notation (3), we may write this as

$$\mathcal{F}^K(s) = L_x L_y \cdot \frac{\hbar\pi c_0}{2L_z^4} \sum_{n_z=0}^{\infty} \int_{r_x=0}^{\infty} dr_x \int_{r_y=0}^{\infty} dr_y \frac{n_z^2}{\sqrt{r_x^2 + r_y^2 + n_z^2}} W_s \left( \sqrt{r_x^2 + r_y^2 + n_z^2} \right).$$

Introducing a polar coordinate representation ( $r_x = R \sin \theta, r_y = R \cos \theta$ ) with  $R^2 = r_x^2 + r_y^2$  and  $dr_x dr_y = R dR d\theta$  we obtain

$$\begin{aligned} \mathcal{F}^K(s) &= L_x L_y \cdot \frac{\hbar\pi c_0}{2L_z^4} \sum_{n_z=0}^{\infty} \int_{R=0}^{\infty} \int_{\theta=0}^{\pi/2} R dR d\theta \frac{n_z^2}{\sqrt{R^2 + n_z^2}} W_s \left( \sqrt{R^2 + n_z^2} \right) \\ &= L_x L_y \cdot \frac{\hbar\pi^2 c_0}{4L_z^6} \sum_{n_z=0}^{\infty} \int_{R=0}^{\infty} R dR \frac{n_z^2}{\sqrt{R^2 + n_z^2}} W_s \left( \sqrt{R^2 + n_z^2} \right). \end{aligned}$$

Introducing the change of variable  $U^2 = R^2 + n_z^2$  with  $U dU = R dR$ , we have

$$\frac{\mathcal{F}^K(s)}{L_x L_y} = \frac{\hbar\pi^2 c_0}{4L_z^4} \sum_{n_z=0}^{\infty} \int_{U=n_z}^{\infty} n_z^2 W_s(U) dU = \frac{\hbar\pi^2 c_0}{4L_z^4} \sum_{n_z=0}^{\infty} g_s(n_z)$$

where

$$g_s(n_z) \equiv \int_{U=n_z}^{\infty} n_z^2 W_s(U) dU = \int_{U=n_z}^{\infty} n_z^2 e^{-sU} dU,$$

where we have chosen the same regulator as before:  $W_s(U) = e^{-sU}$ . This is now in a form suitable to use Euler-Maclaurin (1). With  $m = 2$ , we have

$$\begin{aligned} \sum_{n_z=0}^{\infty} g_s(n_z) - \int_0^{\infty} g_s(x) dx &= \frac{1}{2} \left( g_s(0) + g_s(\infty) \right) + \frac{B_2}{2} (g'_s(\infty) - g'_s(0)) + \frac{B_4}{24} (g'''_s(\infty) - g'''_s(0)) \\ &\quad + \frac{1}{120} \int_0^{\infty} P_5(x) g_s^{(5)}(x) dx \\ &= -\frac{1}{120} + O(s^2) \\ &= -\frac{1}{120} + O(s^2) \end{aligned}$$

using

$$\frac{d}{da} g_s(a) = \frac{d}{da} \int_a^{\infty} a^2 e^{-sU} dU = \int_a^{\infty} 2a e^{-sU} dU - a^2 e^{-sa}$$

and

$$\int_{U=0}^{\infty} e^{-sU} dU = -\frac{e^{-sU}}{s} \Big|_{U=0}^{\infty} = \frac{1}{s}.$$

The only non-zero terms arise from  $-B_4/24 g'''(0)$  and the term involving the periodic Bernoulli functions, though these terms are of order  $s^2$  and so vanish upon renormalization. Thus, we define the regularized total force density (pressure) to be

$$\frac{\mathcal{F}_{\text{reg}}^K(s)}{L_x L_y} = \frac{\hbar \pi^2 c_0}{4L_z^4} \left( \sum_{n_z=0}^{\infty} g_s(n_z) - \int_{n_z=0}^{\infty} g_s(x) dx \right)$$

and the renormalized total force density (pressure) to be

$$\mathcal{P}_{\text{ren}}^K \equiv \frac{\mathcal{F}_{\text{reg}}^K(0)}{L_x L_y} = -\frac{\hbar \pi^2 c_0}{480 L_z^4}.$$

This is only for the TE modes, but there is an analogous derivation to find the renormalized pressure for the TM modes, which is equal to that of the TE modes. Hence, the total pressure is

$$\mathcal{P}_{\text{TOT}}^K = 2\mathcal{P}_{\text{ren}}^K = -\frac{\hbar \pi^2 c_0}{240 L_z^4}.$$

## A. Vacuum Expectation Values

Let  $\hat{\Lambda}_{NM}$  denote an operator-valued  $p$ -form quadratic in the creation and annihilation operators  $\hat{a}_N, \hat{a}_M^\dagger$ . Specifically, let

$$\hat{\Lambda}_{NM} = \hat{\alpha}_N \wedge \hat{\beta}_M$$

in terms of the operator-valued forms

$$\hat{\alpha}_N = \left( \hat{a}_N e^{i\omega_N t} + \hat{a}_N^\dagger e^{-i\omega_N t} \right) \alpha_N \quad \text{and} \quad \hat{\beta}_M = \left( \hat{a}_M e^{i\omega_M t} + \hat{a}_M^\dagger e^{-i\omega_M t} \right) \beta_M$$

where  $\alpha_N, \beta_M$  are differential forms (whose degrees sum to  $p$ ). Then

$$\begin{aligned} \hat{\Lambda}_{NM} &= \left( \hat{a}_N e^{i\omega_N t} + \hat{a}_N^\dagger e^{-i\omega_N t} \right) \left( \hat{a}_M e^{i\omega_M t} + \hat{a}_M^\dagger e^{-i\omega_M t} \right) \alpha_N \wedge \beta_M \\ &= \left( \hat{a}_N \hat{a}_M e^{i(\omega_N + \omega_M)t} + \hat{a}_N^\dagger \hat{a}_M e^{-i(\omega_N - \omega_M)t} + \hat{a}_N \hat{a}_M^\dagger e^{i(\omega_N - \omega_M)t} + \hat{a}_N^\dagger \hat{a}_M^\dagger e^{-i(\omega_N + \omega_M)t} \right) \alpha_N \wedge \beta_M. \end{aligned}$$

From (5), we have

$$\begin{aligned} \langle 0 | \hat{a}_N \hat{a}_M | 0 \rangle &= 0 & \langle 0 | \hat{a}_N^\dagger \hat{a}_M | 0 \rangle &= 0, \\ \langle 0 | \hat{a}_N \hat{a}_M^\dagger | 0 \rangle &= \delta_{NM} & \langle 0 | \hat{a}_N^\dagger \hat{a}_M^\dagger | 0 \rangle &= 0 \end{aligned}$$

where we have used the commutation relation (5) and the normalization of the vacuum state  $\langle 0 | 0 \rangle = 1$ . This gives

$$\langle 0 | \hat{\Lambda}_{NM} | 0 \rangle = \delta_{NM} e^{i(\omega_N - \omega_M)t} \alpha_N \wedge \beta_M = \alpha_N \wedge \beta_M.$$